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Energy transfer model and large periodic boundary value problem for the quintic NLS

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1 Introduction

This note is based on a talk given at the conference on Nonlinear Wave and Dispersive Equations, Kyoto University. We consider a dynamics of mass density $|\widehat{u}(t,\xi)|$ and energy exchanges between a linear oscillator and a nonlinear interaction for the following defocusing quintic NLS equation:

$$i\partial_t u+\partial_x^2 u=|u|^4u, \quad t\in\mathbb{R},\ x\in\mathbb{T}_L=[0,2\pi L], \tag{1.1}$$

where $u=u(t,x)\,:\,\mathbb{R}\times\mathbb{T}_L\to\mathbb{C}$ is a complex-valued function and the spatial domain $\mathbb{T}_L$ is taken to be a torus of length $2\pi L>0$, i.e., we assume the periodic boundary condition:

$$u(t,x+2\pi L)=u(t,x).$$

The aim is to understand the dynamics of mass density $|\widehat{u}(t,\xi)|$, namely,

(i) how the wave energy is exchanged to another,

(ii) provide a demonstration of the conservative energy exchange of solutions between the modes initially excited.

The equation (1.1) possesses at least two conservation laws, the mass $M[u](t)$ and energy $E[u](t)$:

$$M[u](t)=\|u(t)\|_{L^2}^2,$$
$$E[u](t)=\int_{\mathbb{T}_L}\frac{1}{2}|\partial_x u(t,x)|^2+\frac{1}{6}|u(t,x)|^6\,dx=E[u](0).$$

These quantities impose the constraints on a dynamics of mass density of solutions.

We expect the following conjecture.

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Conjecture 1.1. *On bounded domain case: if the nonlinear Schrödinger equation is not integrable, then there exists a time global smooth solution $u(t)$ and some Sobolev exponent $s > 1$ such that*

$$\lim_{|t|\to\infty} \|u(t)\|_{H^s} = \infty.$$

Remark 1.1. The above conjecture means that the mass is shifted to high frequencies.

In fact, on $\mathbb{R}$ domain case, the local well-posedness for (1.1) was proved by Cazenave and Weissler [3] for data in L^2 (see also [8] and [12]). Notice that the time of existence time depends on the position of data and not only on its size. One can also prove the global well-posedness in L^2 provided that the initial data in L^2 is sufficiently small by using the above conservation laws. It is should be noted that the equation (1.1) is left invariant by the scaling

$$u \mapsto u_\lambda; \quad u(t,x) \mapsto u_\lambda(t,x) = \lambda^{1/2}u(\lambda^2 t, \lambda x), \qquad \lambda > 0,$$

which preserves the homogeneous Sobolev norm $\dot{H}^s(\mathbb{R})$ with $s = 0$. The global existence result for any data in L^2 was proved by Dodson [6]. In [6], the deep results on scattering behavior of solutions were also obtained. On the other hand, the associated focusing nonlinear Schrödinger equation (the minus sign applies to the nonlinear term) has a finite time blow-up solution [9].

2 Past works on the periodic boundary domain

When the spatial dimension is the two-dimensional torus $\mathbb{T}^2$, Bourgain [2] considered the cubic nonlinear Schrödinger equation in the defocusing case

$$i\partial_t u + \Delta u = |u|^2 u,$$

and obtained the apriori estimate of solutions

$$\|u(t)\|_{H^s} \lesssim \langle t\rangle^{2(s-1)+} \quad \text{for } u(0) \in H^s, \ s \geq 4.$$

In [5], Colliander, Keel, Staffilani, Takaoka and Tao constructed the solution satisfying that for any $s > 1$, $K \gg 1$, $0 < \sigma < 1$ there exist a solution $u(t)$ and a time $T > 0$ such that

$$\|u(0)\|_{H^s} \leq \sigma \quad \text{and} \quad \|u(T)\|_{H^s} \geq K.$$

Observe that the cubic nonlinear Schrödinger equation in two spatial dimensions is known as an example of invariant under the L^2-scaling:

$$u \mapsto u_\lambda = \lambda u(\lambda^2 t, \lambda x) \quad (\lambda > 0).$$

On the other hand, the quintic nonlinear Schrödinger equation in one dimension obeys scale invariance under the L^2-scaling. In [10], Grébert and L. Thomann examined the dynamics exhibited by the following Cauchy problem

$$i\partial_t u + \partial_x^2 u = \nu|u|^4 u, \quad (t,x) \in \mathbb{R}\times\mathbb{T}, \tag{2.1}$$

for $\nu > 0$. More precisely, they proved the following theorem.

Theorem 2.1 (Grébert and Thomann [10]). *Let $k \in \mathbb{Z}\backslash\{0\}$ and $n \in \mathbb{Z}$. $\mathcal{A}$ is a set of the form $\mathcal{A} = \{a_2,\ a_1,\ b_2,\ b_1\}$ where $a_2 = n,\ a_1 = n+3k,\ b_2 = n+4k,\ b_1 = n+k$. There exist $T > 0,\ \lambda_0 > 0$ and a $2T$-periodic function $K_* : \mathbb{R} \mapsto (0,1)$ which satisfies $K_*(0) \leq 1/4$ and $K_*(T) \geq 3/4$ so that if $0 < \nu < \nu_0$, there exists a solution to (2.1) satisfying for all $0 \leq t \leq \nu^{-3/2}$*

$$u(t,x) = \sum_{j\in\mathcal{A}} u_j(t)e^{ijx} + \nu^{1/4}q_1(t,x) + \nu^{3/2}tq_2(t,x),$$

with

$$|u_{a_1}(t)|^2 = 2|u_{a_2}(t)|^2 = K_*(\nu t),$$
$$|u_{b_1}(t)|^2 = 2|u_{b_2}(t)|^2 = 1 - K_*(\nu t),$$

and where for all $s \in \mathbb{R}$, $\|q_1(t,\cdot)\|_{H^s(\mathbb{T}_{\mathbb{Z}})} \leq C_s$ for all $t \in \mathbb{R}_+$, and $\|q_2(t,\cdot)\|_{H^s(\mathbb{T})} \leq C_s$ for all $0 \leq t \leq \nu^{-3/2}$.

We now define the wavenumber set consisting of nonlinear resonance interactions in the equation (1.1).

Definition 2.1 (Resonance interaction set). Let $n \in \mathbb{Z}$ and $k \in \mathbb{Z}\backslash\{0\}$ be fixed. For $j \in \mathbb{Z}$, we set $\alpha_{1,j},\ \alpha_{3,j},\ \alpha_{2,j}$ and $\alpha_{4,j}$ as follows:

$$\frac{\alpha_{1,j}}{2\pi} = n + 3k + \frac{j}{L},\ \frac{\alpha_{3,j}}{2\pi} = n + \frac{j}{L},\ \frac{\alpha_{2,j}}{2\pi} = n + k + \frac{j}{L},\ \frac{\alpha_{4,j}}{2\pi} = n + 4k + \frac{j}{L}.$$

With $\alpha_{m,j}$, we set

$$\mathcal{A}_m = \{\alpha_{m,j} \mid j \in \mathbb{Z},\ 0 \leq j \leq L\},$$

for $1 \leq m \leq 4$, and $\mathcal{C} = \cup_{m=1}^4 \mathcal{A}_m$.

Here we consider (1.1) on $\mathbb{T}_L$ instead of (2.1) on $\mathbb{T}$, and obtain the following theorem.

Theorem 2.2. *Let $n \in \mathbb{Z},\ k \in \mathbb{Z}\backslash\{0\},\ s \geq 1$ and large integer $L > 0$ be given. There exist a smooth global solution $u(t)$ and a time $T = O(L^{1/2-})$ s.t.*

$$u(t,x) = \sum_{m=1}^{4} u_{\mathcal{A}_m}(t,x) + e(t,x),$$

where

$$\|u_{\mathcal{A}_1}(t)\|_{L^2}^2 = 2\|u_{\mathcal{A}_3}(t)\|_{L^2}^2 = \frac{1}{2} - K(t),$$
$$\|u_{\mathcal{A}_2}(t)\|_{L^2}^2 = 2\|u_{\mathcal{A}_4}(t)\|_{L^2}^2 = \frac{1}{2} + K(t),$$

$K(t) \approx \sin(\mathrm{Arctan}\frac{t}{4L^3})$ *for* $|t| \lesssim L^{1/2-}$,

$$\sup_{|t|\leq T} \|e(t,\cdot)\|_{H^s} \lesssim \frac{1}{L^{1/2-}}.$$

where a constant c_j's are independent of L.

Remark 2.1. Putting $n = 0,\ s \geq 0,\ |k|^s \gg L^{\frac{5}{2}+\varepsilon}$, we have that there exists a solution $u(t)$ s.t. for some time $t_0 > 0$,

$$\|u(t_0)\|_{H^s}^2 - \|u(-t_0)\|_{H^s}^2 \approx \langle k\rangle^{2s}(1 + \frac{4^{2s}}{2} - 3^{2s})K(t_0).$$

If $s > 1$, then $\|u(-t_0)\|_{H^s} < \|u(t_0)\|_{H^s}$ (energy does not decrease). As opposites, we can construct solution whose energy will not increase for a long time.

3 Proof of Theorem 2.2

The proof of Theorem 2.2 consists of three steps:

1. construct resonant sets,
2. construct finite dimensional model with initial replacement,
3. construct approximation lemma (error estimates).

3.1 Resonant sets

We first set nonlinear resonant interaction sets as follows.

Definition 3.1. We say the set $\{(\xi_1, \xi_3, \xi_5),\ (\xi_2, \xi_4, \xi_6)\}$ is resonance, if and only if the following conditions hold;

(i) $\xi_1 + \xi_3 + \xi_5 = \xi_2 + \xi_4 + \xi_6$,

(ii) two of ξ_1, ξ_3, ξ_5 are elements of $\mathcal{A}_1$, that are $n + 3k + \frac{j_1}{L},\ n + 3k + \frac{j_3}{L}$,

(iii) one of ξ_1, ξ_3, ξ_5 is element of $\mathcal{A}_2$, that is $n + \frac{j_5}{L}$,

(iv) two of ξ_2, ξ_4, ξ_6 are elements of $\mathcal{A}_3$, that are $n + k + \frac{j_2}{L},\ n + k + \frac{j_4}{L}$,

(v) one of ξ_2, ξ_4, ξ_6 is element of $\mathcal{A}_4$, that is $n + 4k + \frac{j_6}{L}$,

(vi) $\{j_1, j_3\} = \{j_2, j_4\}$ and $j_5 = j_6 = \frac{j_1+j_3}{2} = \frac{j_2+j_4}{2}$.

From the relation in the above definition, we have the following lemma.

Lemma 3.1. *If* $\{(\xi_1, \xi_3, \xi_5),\ (\xi_2, \xi_4, \xi_6)\}$ *is resonance, then* $\phi(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) = 0$.

Proof. Since the equation $(n + 3k)^2 + (n + 3k)^2 + n^2 = (n + k)^2 + (n + k)^2 + (n + 4k)^2$, we calculate

$$\frac{1}{(2\pi)^2}\phi(\xi_1, \xi_2.\xi_3, \xi_4, \xi_5, \xi_6) = \frac{2k}{L}(3(j_1 + j_3) - (j_2 + j_4) - 4j_6) + \frac{1}{L^2}\phi(j_1, j_2, j_3, j_4, j_5, j_6),$$

which is equal to zero by $j_6 = \frac{j_1+j_3}{2} = \frac{j_2+j_4}{2}$ and $\phi(j_1, j_2, j_3, j_4, j_5, j_6) = 0$. $\square$

3.2 Finite dimensional model

Now suppose that u is a smooth solution and let us start with the ansatz

$$u(t,x) = e^{-iGt} \int a_\xi(t) e^{ix\xi - it\xi^2} \, (d\xi)_L,$$

where

$$\phi(x) = \int e^{ix\xi} \widehat{\phi}(\xi) \, (d\xi)_L := \frac{1}{L} \sum_{\xi \in 2\pi\mathbb{Z}/L} e^{ix\xi} \widehat{\phi}(\xi).$$

We choose the parameter $G = -\frac{6M_0^2}{L^2}$ where $M_0 = \|u(0)\|_{L^2}^2$. By direct calculation, $a_\xi = a_\xi(t)$ satisfies

$$\begin{aligned} \dot{a}_\xi &= \left(-\frac{6M_0}{L^3}|a_\xi|^2 - \frac{3}{L^3} \int |a_{\xi'}|^4 (d\xi')_L + \frac{4}{L^4}|a_\xi|^4 \right) a_\xi \\ &\quad + \int_{\{\xi_1,\xi_3,\xi_5\} \overset{*}{\neq} \{\xi_2,\xi_4,\xi\}} a_{\xi_1} \overline{a_{\xi_2}} a_{\xi_3} \overline{a_{\xi_4}} a_{\xi_5} e^{it\phi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi)}, \end{aligned} \tag{3.1}$$

where $\phi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = -\xi_1^2 + \xi_2^2 - \xi_3^2 + \xi_4^2 - \xi_5^2 + \xi_6^2$.

Then plugging the observation in Lemma 3.1 into the equation (3.1), we have the resonant formula

$$\begin{aligned} i\dot{r}_\xi &= \left(-\frac{6M_0}{L^3}|r_\xi|^2 - \frac{3}{L^3} \int |r_{\xi'}|^4 (d\xi')_L + \frac{4}{L^4}|r_\xi|^4 \right) r_\xi \\ &\quad + \int_{\mathrm{res}(\xi)} r_{\xi_1} \overline{r_{\xi_2}} r_{\xi_3} \overline{r_{\xi_4}} r_{\xi_5}, \end{aligned} \tag{3.2}$$

where $\mathrm{res}(\xi)$ denotes the resonant modes with respect to ξ_j, $1 \le j \le 5$ such that the set $\{(\xi_1,\xi_3,\xi_5),(\xi_2,\xi_4,\xi)\}$ is resonance.

3.3 A priori estimates

Next, observe the conserved quantities for (3.1).

Lemma 3.2. *Let $\{r_\xi(t)\}$ be a global solution to recasted NLS (3.2). Then we have the relations;*

$$\frac{d}{dt} \sum_{\xi \in C} |r_\xi(t)|^2 = 0, \tag{3.3}$$

$$\frac{d}{dt} \sum_{\xi \in C} |\xi|^2 |r_\xi(t)|^2 = 0. \tag{3.4}$$

Remark 3.1. These corresponds to the mass and the energy conservations, respectively.

Proof of Lemma 3.2. It will be convenience to change the index ξ by ξ_6. We first prove (3.3). Multiplying $\overline{r_{\xi_6}}$ to (3.2) and taking the imaginary part, we have that

$$\mathrm{Im}(i\dot{r}_{\xi_6}\overline{r_{\xi_6}}) = \frac{1}{L^4}\mathrm{Im}\sum_{\mathrm{res}(\xi_6)} r_{\xi_1}\overline{r_{\xi_2}}r_{\xi_3}\overline{r_{\xi_4}}r_{\xi_5}\overline{r_{\xi_6}}.$$

Note that the left-hand side will be $-\frac{1}{2}\frac{d}{dt}|r_{\xi_6}|^2$. Then after the summation over the resonant set, we arrive at the following;

$$-\frac{1}{2}\frac{d}{dt}\sum_{\xi_6\in\mathcal{C}}|r_{\xi_6}(t)|^2 = \frac{1}{2iL^4}\sum_{\xi_6\in\mathcal{C}}\sum_{\mathrm{res}(\xi_6)}\left(r_{\xi_1}\overline{r_{\xi_2}}r_{\xi_3}\overline{r_{\xi_4}}r_{\xi_5}\overline{r_{\xi_6}} - \overline{r_{\xi_1}}r_{\xi_2}\overline{r_{\xi_3}}r_{\xi_4}\overline{r_{\xi_5}}r_{\xi_6}\right),$$

which is zero, since by symmetry.

Next we prove (3.4). In a similar way to above, we have

$$\begin{aligned}\frac{d}{dt}\sum_{\xi\in\mathcal{C}}|\xi|^2|r_\xi(t)|^2 &= 2\mathrm{Re}\sum_{\xi_6\in\mathcal{C}}|\xi_6|^2\dot{r}_{\xi_6}\overline{r_{\xi_6}}\\ &= \frac{2}{L^4}\mathrm{Im}\sum_{\xi_6\in\mathcal{C}}\sum_{\mathrm{res}(\xi_6)}|\xi_6|^2 r_{\xi_1}\overline{r_{\xi_2}}r_{\xi_3}\overline{r_{\xi_4}}r_{\xi_5}\overline{r_{\xi_6}},\end{aligned}$$

which is deduced by

$$-\frac{1}{3iL^4}\sum_{\mathrm{res}}\left((\xi_1^2+\xi_3^2+\xi_5^2)-(\xi_2^2+\xi_4^2+\xi_6^2)\right) r_{\xi_1}\overline{r_{\xi_2}}r_{\xi_3}\overline{r_{\xi_4}}r_{\xi_5}\overline{r_{\xi_6}},$$

since by symmetrization. Th e last term is zero, since $(\xi_1^2+\xi_3^2+\xi_5^2)-(\xi_2^2+\xi_4^2+\xi_6^2)=0$ for the resonance interaction modes. □

Let us consider the ansatz

$$r_\xi(t) = \sqrt{I_\xi(t)}e^{i\theta_\xi(t)}.$$

Once again, using this coordinate, we obtain the following lemma.

Lemma 3.3. *For $j\in[0,L]$, we have that*

$$\frac{d}{dt}\left(I_{n+\frac{j}{L}}(t) + I_{n+4k+\frac{j}{L}}(t)\right) = 0,$$

$$\frac{d}{dt}\left(I_{n+3k+\frac{j}{L}}(t) + I_{n+k+\frac{j}{L}}(t)\right) = 0,$$

Moreover, assuming additional constrains of $I_j(t)$ and $\theta_j(t)$, we have the following lemma.

Lemma 3.4. *Assume*

$$I_{n+\frac{j}{L}}(t) = I_{n+\frac{m}{L}}(t), \quad \theta_{n+\frac{j}{L}}(t) = \theta_{n+\frac{m}{L}}(t),$$

$$I_{n+3k+\frac{j}{L}}(t) = I_{n+3k+\frac{m}{L}}(t), \quad \theta_{n+3k+\frac{j}{L}}(t) = \theta_{n+3k+\frac{m}{L}}(t),$$
$$I_{n+4k+\frac{j}{L}}(t) = I_{n+4k+\frac{m}{L}}(t), \quad \theta_{n+4k+\frac{j}{L}}(t) = \theta_{n+4k+\frac{m}{L}}(t),$$
$$I_{n+k+\frac{j}{L}}(t) = I_{n+k+\frac{m}{L}}(t), \quad \theta_{n+k+\frac{j}{L}}(t) = \theta_{n+k+\frac{m}{L}}(t)$$

for all $|j|, |m| \leq L$. *Then*

$$\frac{d}{dt}\sum\left(I_{n+3k+\frac{j}{L}}(t) - 2I_{n+\frac{j}{L}}(t)\right) = 0,$$

$$\frac{d}{dt}\sum\left(I_{n+k+\frac{j}{L}}(t) - 2I_{n+4k+\frac{j}{L}}(t)\right) = 0.$$

Proof of Lemmas 3.3 and 3.4. The proof of Lemmas 3.3 and 3.4 is similar to the one of Lemma 3.2. □

Then by Lemmas 3.2, 3.3 and 3.4, it is reasonable that we recast the equation (3.2) into the following Toy model form:

$$\dot{I}_{\mathcal{A}_2} = \frac{3!L^2 + 3(2L+1)}{(2L+1)L^4}\sqrt{I^2_{\mathcal{A}_2}I_{\mathcal{A}_4}I^2_{\mathcal{A}_1}I_{\mathcal{A}_3}}\sin\left(2\theta_{\mathcal{A}_2} + \theta_{\mathcal{A}_4} - 2\theta_{\mathcal{A}_1} - \theta_{\mathcal{A}_3}\right),$$

$$\dot{I}_{\mathcal{A}_1}(t) = \frac{3!(2L^2 + 2L+1)}{(2L+1)L^4}\sqrt{I^2_{\mathcal{A}_2}I_{\mathcal{A}_4}I^2_{\mathcal{A}_1}I_{\mathcal{A}_3}}\sin\left(2\theta_{\mathcal{A}_2} + \theta_{\mathcal{A}_4} - 2\theta_{\mathcal{A}_1} - \theta_{\mathcal{A}_3}\right),$$

$$\dot{I}_{\mathcal{A}_4} = \frac{3!L^2 + 3(2L+1)}{(2L+1)L^4}\sqrt{I^2_{\mathcal{A}_1}I_{\mathcal{A}_3}I^2_{\mathcal{A}_2}I_{\mathcal{A}_4}}\sin\left(2\theta_{\mathcal{A}_1} + \theta_{\mathcal{A}_3} - 2\theta_{\mathcal{A}_2} - \theta_{\mathcal{A}_4}\right)$$

$$\dot{I}_{\mathcal{A}_2}(t) = \frac{3!(2L^2 + 2L+1)}{(2L+1)L^4}\sqrt{I^2_{\mathcal{A}_1}I_{\mathcal{A}_3}I^2_{\mathcal{A}_2}I_{\mathcal{A}_4}}\sin\left(2\theta_{\mathcal{A}_1} + \theta_{\mathcal{A}_3} - 2\theta_{\mathcal{A}_2} - \theta_{\mathcal{A}_4}\right),$$

$$\begin{aligned}-\dot{\theta}_{\mathcal{A}_3} &= -\frac{6M_0}{L^3}I_{\mathcal{A}_1} - \frac{12(L+1)}{L^4}\left(I_{\mathcal{A}_3} + I_{\mathcal{A}_1} + I_{\mathcal{A}_4} + I_{\mathcal{A}_2}\right) + \frac{4}{L^4}I^2_{\mathcal{A}_13}\\ &\quad + \frac{3!L^2 + 3(2L+1)}{(2L+1)L^4}\sqrt{\frac{I^2_{\mathcal{A}_2}I_{\mathcal{A}_4}I^2_{\mathcal{A}_1}}{I_{\mathcal{A}_3}}}\cos\left(2\theta_{\mathcal{A}_2} + \theta_{\mathcal{A}_4} - 2\theta_{\mathcal{A}_1} - \theta_{\mathcal{A}_3}\right),\end{aligned}$$

$$\begin{aligned}-\dot{\theta}_{\mathcal{A}_1} &= -\frac{6M_0}{L^3}I_{\mathcal{A}_1} - \frac{12(L+1)}{L^4}\left(I_{\mathcal{A}3} + I_{\mathcal{A}_1} + I_{\mathcal{A}_4} + I_{\mathcal{A}_2}\right) + \frac{4}{L^4}I^2_{\mathcal{A}_1}\\ &\quad + \frac{3!(2L^2 + 2L+1)}{(2L+1)L^4}\sqrt{I^2_{\mathcal{A}_2}I_{\mathcal{A}_4}I_{\mathcal{A}_3}}\cos\left(2\theta_{\mathcal{A}_2} + \theta_{\mathcal{A}_4} - 2\theta_{\mathcal{A}_1} - \theta_{\mathcal{A}_3}\right),\end{aligned}$$

$$
\begin{aligned}
-\dot{\theta}_{\mathcal{A}_4} &= -\frac{6M_0}{L^3} I_{\mathcal{A}_4} - \frac{12(L+1)}{L^4}\left(I_{\mathcal{A}_3} + I_{\mathcal{A}_1} + I_{\mathcal{A}_4} + I_{\mathcal{A}_2}\right) + \frac{4}{L^4} I_{\mathcal{A}_4}^2 \\
&\quad + \frac{3!L^2 + 3(2L+1)}{(2L+1)L^4} \sqrt{\frac{I_{\mathcal{A}_1}^2 I_{\mathcal{A}_3} I_{\mathcal{A}_2}^2}{I_{\mathcal{A}_4}}} \cos\left(2\theta_{\mathcal{A}_2} + \theta_{\mathcal{A}_4} - 2\theta_{\mathcal{A}_1} - \theta_{\mathcal{A}_3}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
-\dot{\theta}_{\mathcal{A}_2} &= -\frac{6M_0}{L^3} I_{\mathcal{A}_2} - \frac{12(L+1)}{L^4}\left(I_{\mathcal{A}_3} + I_{\mathcal{A}_1} + I_{\mathcal{A}_4} + I_{\mathcal{A}_2}\right) + \frac{4}{L^4} I_{\mathcal{A}_2}^2 \\
&\quad + \frac{3!(2L^2 + 2L+1)}{(2L+1)L^4} \sqrt{I_{\mathcal{A}_1}^2 I_{\mathcal{A}_3} I_{\mathcal{A}_4}} \cos\left(2\theta_{\mathcal{A}_2} + \theta_{\mathcal{A}_4} - 2\theta_{\mathcal{A}_1} - \theta_{\mathcal{A}_3}\right).
\end{aligned}
$$

Remark 3.2. A straightforward calculation show that this ODE system enjoys the conservation of the Hamiltonian following Hamiltonian

$$
\begin{aligned}
H &= -\frac{3M_0}{L^3}\left(I_{\mathcal{A}_1}^2 + I_{\mathcal{A}_2}^2 + I_{\mathcal{A}_3}^2 + I_{\mathcal{A}_4}^2\right) - \frac{6(L+1)}{L^4}\left(I_{\mathcal{A}_1} + I_{\mathcal{A}_2} + I_{\mathcal{B}_1} + I_{\mathcal{B}_2}\right)^2 \\
&\quad + \frac{4}{3L^4}\left(I_{\mathcal{A}_1}^3 + I_{\mathcal{A}_2}^3 + I_{\mathcal{A}_3}^3 + I_{\mathcal{A}_4}^3\right) \\
&\quad + \frac{3!(2L^2+2L+1)}{(2L+1)L^4} I_{\mathcal{A}_1} I_{\mathcal{A}_2} I_{\mathcal{A}_3}^{\frac{1}{2}} I_{\mathcal{A}_4}^{\frac{1}{2}} \cos\left(2\theta_{\mathcal{A}_2} + \theta_{\mathcal{A}_4} - 2\theta_{\mathcal{A}_1} - \theta_{\mathcal{A}_3}\right).
\end{aligned}
$$

Indeed, we see that

$$
\begin{cases}
\dot{\theta}_{\mathcal{C}} = -\frac{\partial H}{\partial I_{\mathcal{C}}}, \\
\dot{I}_{\mathcal{C}} = \frac{\partial H}{\partial \theta_{\mathcal{C}}},
\end{cases}
$$

for $\mathcal{C} = \mathcal{A}_j$.

In virtue of $\frac{d}{dt}(2I_{\mathcal{A}_4}^k + I_{\mathcal{A}_3}^k - 2I_{\mathcal{A}_2}^k - I_{\mathcal{A}_1}^k) = 0$ for $k = 1, 2$, we obtain the particular dynamics of $I_{\mathcal{A}_j}$, $\theta_{\mathcal{A}_j}$.

Proposition 3.1. *There exists a solution* $(I_{\mathcal{A}_j}, \theta_{\mathcal{A}_j})_{1\le j\le 4}$ *to (Toy model) s.t.*

$$
2\theta_{\mathcal{A}_2} + \theta_{\mathcal{A}_4} - 2\theta_{\mathcal{A}_1} - \theta_{\mathcal{A}_3} \approx \frac{\pi}{2}
$$

and

$$
I_{\mathcal{A}_1}(t) = 2I_{\mathcal{A}_3}(t) = \frac{1}{2} - K(t),\ I_{\mathcal{A}_2}(t) = 2I_{\mathcal{A}_4}(t) = \frac{1}{2} + K(t),
$$

$K(t) \approx \sin(\mathrm{Arctan}\frac{t}{4L^3})$

Proof of Proposition 3.1. The proof uses the symplectic change of variables.

3.4 Approximation lemma

We show how the toy model approximates the original NLS.

Proposition 3.2. *Let $\{a_\xi(t)\}_{\xi\in\mathbb{Z}/L}$ and $\{r_\xi(t)\}_{\xi\in\mathcal{C}}$ be a solution to the Fourier transformed NLS equation and its resonant NLS, respectively, with $a_\xi(0) = r_\xi(0)$ for $\xi \in \mathcal{C}$ and $\|a_\xi(0)\|_{\ell^2_s(\xi\notin\mathcal{C})} \lesssim \frac{1}{L^{1/2-\varepsilon}}$. Then for $s \geq 1$ and $|t| \leq cL^{1/2-\varepsilon}$,*

$$\|a_\xi(t) - r_\xi(t)\|_{\ell^2_s} \lesssim \frac{1}{L^{1/2-\varepsilon}},$$

where $r_\xi(t) = 0$ if $\xi \notin \mathcal{C}$, and $\|f\|_{\ell^2_s} = \|\langle\xi\rangle^s f(\xi)\|_{\ell^2}$.

Proof of Proposition 3.2. The proof uses a priori estimates of $M[u]$, $E[u]$, energy argument, bootstrap argument and the perturbation argument. □

By Propositions 3.1 and 3.2, we conclude the proof of Theorem 2.2.

References

[1] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, part I: Schrödinger equation, part II: The KdV-equation*, Geom. Func. Anal., **3** (1993), 107–156, 209–262.

[2] J. Bourgain, *On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE*, Internat. Math. Res. Notices, 1996, 277–304.

[3] T. Cazenave and F. B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Anal., **14** (1990), 807–836.

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$*, J. Amer. Math. Soc., **16** (2003), 705–749.

[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation*, Invent. Math., **181** (2010), 39–113.

[6] B. Dodson, *Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schödinger equation when $d = 1$*, Amer. J. Math., **138** (2016), 531–569.

[7] E. Faou, P. Germain and Z. Hani, *The weakly nonlinear large-box limit of the 2d cubic nonlinear Schrödinger equation*, J. Amer. Math. Soc., **29** (2015), 915–982.

[8] J. Ginibre and G. Velo, *On the class of nonlinear Schrödinger equations*, J. Funct. Anal., **32** (1979), 1–32, 33–72.

[9] R. T. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys., **18** (1977), 1794–1797.

[10] B. Grébert and L. Thomann, *Resonant dynamics for the quintic nonlinear Schrödinger equation*, Ann. I. H. Poincaré, **29** (2012), 455–477.

[11] H. Takaoka, *Energy transfer model for the derivative nonlinear Schrödinger equations on the torus*, Discrete Contin. Dyn. Syst., **37** (2017), 5819–5841.

[12] Y. Tsutsumi, L^2*-solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcialaj Ekvacioj, **30** (1987), 115- 125.

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